Comment on “Scaling of random fields by means of truncated power variograms and associated spectra”
by Vittorio Di Federico and Shlomo P. Neuman

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In the past it has been common to assume that the log conductivity \([\log(K)]\) distribution in geological media is characterized by a statistically homogeneous random field. However, observations have shown that the variance and correlation scale of real \(\log(K)\) fields increase with domain size, even though they may appear statistically homogeneous at a given support scale. The recent paper by Di Federico and Neuman [1997] develops a hierarchical theory of \(\log(K)\) distributions to interpret the observed general relationships between measurements and measurement scale. In this comment we do not intend to debate the main conclusions of their excellent paper. Rather, we would like to provide a different perspective on the derivation of two of their relationships and discuss what appears to be a minor, but possibly important, inconsistency between the underlying theory and the result of fitting the theory to data.

Like Di Federico and Neuman [1997], we assume that a \(\log(K)\) distribution is generally characterized by an isotropic fractional Brownian motion (fBm) with stationary increments having the usual power law variogram given by

\[
\gamma(h) = C_s h^{2H} \quad 0 < H < 1 \tag{1}
\]

where \(h\) is lag, and a vector for two- or three-dimensional problems; \(h = |h|; \) \(C_s\) is a constant; and \(H\) is the Hurst coefficient. Such a variogram grows in an unbounded manner with lag, but when field measurements are being made one is always working within a finite domain, or “window,” of characteristic length \(L\). For a fBm-like process in a finite domain, it is always possible to define an equivalent, statistically homogeneous \(\log(K)\) field. It is equivalent in the sense of having a variogram given by (1) over the problem domain, and it is statistically homogeneous by having a finite correlation scale. Using the well-known relationships between variogram, autocorrelation function, and variance, one can specify the field as in (2), provided that the introduced parameter, \(\eta\), is unity. (The fact that observations do not obviously support a value of unity for \(\eta\) will be discussed shortly.)

\[
C(h) = 0.5C_s (L \eta)^{2H} - h^{2H} \quad h \leq L \eta
\]
\[
C(h) = 0 \quad h > L \eta \tag{2}
\]
\[
\sigma^2 = 0.5C_s (L \eta)^{2H}
\]

In (2), \(C\) is autocovariance, \(\sigma^2\) is variance, and it can be shown easily that the random field defined by (2) has the same variogram as (1) for \(h < L \eta\). The corresponding correlation (integral) scale \(\lambda\) and variance \(\sigma^2\) have the following properties:

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\[
\lambda = \frac{2H}{1 + 2H} L \tag{3}
\]

and

\[
\sigma^2 = \alpha L^{2H} \tag{4}
\]

On the basis of the above brief reasoning, we essentially arrive at the same result as Di Federico and Neuman [1997], as represented by their equations (51) and (52), with their \(\mu\) equivalent formally to our \(\eta\). (The fact that a high-frequency cutoff was not included in our development is immaterial to the following argument.) For comparison with our equation (3), their equations (51) and (52) may be combined to yield

\[
I = \frac{2H}{1 + 2H} L \tag{5}
\]

where \(I\) is the integral scale, which is equivalent to \(\lambda\) in (3), and \(L\) is the same as in (3) and (5). The \(\mu\) in equation (51) of Di Federico and Neuman [1997] (or (5) herein) is introduced to represent the ratio between the length scale \((L)\) of a sampling window and the integral scale of the lowest-frequency mode that is filtered out by that particular window. It is further postulated that \(\mu\) is a constant. Fitting (51) to the data of their Figure 8, Di Federico and Neuman [1997] arrive at a value of about 1/3 for \(\mu\).

The point we wish to make is that strictly speaking, to be consistent with an underlying fBm field defined by our (2), \(\eta\) or \(\mu\) should be 1. We also note that \(\mu\) is not an essential feature of the theory developed by Di Federico and Neuman [1997] but was introduced as a postulate in order to fit their theory to data. Thus the question is open as to why \(\mu\) or \(\eta\) appears to be significantly less than 1 at all scales. We conjecture that non-uniformity may result from at least two considerations:

First, (1) defines a scale-free random field. Real geologic media, however, may not be completely scale-free but have a range of discrete scales. This point was implicit in work by Anderson [1991] and discussed further by Neuman [1994], and may be related to the lacunarity concept developed by Mandelbrot [1983], which describes the uniformity of gaps or voids in a fractal object. The lacunarity concept associated with \(\log(K)\) distributions is yet to be investigated in the literature. If the observed domain is between two adjacent scales such that effects of the relatively large-scale variations can be excluded, a true sill for the variogram may exist under certain conditions and be observed, resulting in \(\eta < 1\). However, this effect would not be expected to occur at virtually all sites (or all scales) and so cannot offer a full explanation for the fact that calculated \(\eta\) values from observations are almost always smaller than 1 [Di Federico and Neuman, 1997, Figure 8].

A second consideration is that the correlation scale, \(\lambda\), to which \(\eta\) is proportional for a given \(L\), may be underestimated.
systematically in the relevant estimation procedures. For example, it is a common practice to estimate a variogram from data pairs with lags less than or equal to 0.5L [e.g., Eggleston et al., 1996] because of a high degree of uncertainty for larger lags. Because \((d\gamma)/(dh) \sim h^{2H-1}\), which decreases rapidly with \(h\) for small \(H\) values, it is not easy to identify a possible trend from noisy data even with lags smaller than 0.5L. It is therefore quite possible one might consider that a sill has been achieved at \(h = 0.5L\) in the estimation procedure. This argument is equivalent to assuming an "observed" \(\eta\) to be about 0.5, although the "true" \(\eta\) value would be 1. In reality, both considerations mentioned above can simultaneously contribute to the observation that estimated \(\eta\) values are generally much smaller than 1.

An important element for understanding and predicting fluid flow and mass transport in the subsurface is to identify and develop a mathematical framework for describing subsurface heterogeneity. The paper of Di Federico and Neuman [1997], together with the relevant work of Neuman [1990, 1994], is an important contribution to this field. Coming from a smaller-scale viewpoint and relying on detailed \(K\) measurements, Molz and Boman [1993, 1995] and Liu and Molz [1996] also showed the existence of different scaling properties in \(K\) distributions. It is evident now that both lines of reasoning are converging on the same general conclusion: that \(K\) distributions exhibit scaling structures that may possibly be used to deal with and understand the scale problem in heterogeneous porous media. At the present time fractals are being used as a model for such scaling. It should be indicated, however, that studies based on detailed \(K\) measurements at relatively small scales are indicating more complicated fractal structures than the isotropic Gaussian fractals assumed by Di Federico and Neuman [1997] and Neuman [1990, 1994]. Measurements suggest that a three-dimensional \(K\) distribution may be characterized by different fractal structures at different sites and at different measurement scales. For example, there is evidence for different Gaussian fractal structures in different directions, or the same fractal structure, but with different \(H\) values. There is support also for non-Gaussian fractal structure [Painter, 1996; Liu and Molz, 1997]. Further study of possibly complicated, anisotropic, fractal structures in \(K\) distributions may offer an improved understanding of the observed relations between correlation scale, variance, domain size, and solute transport behavior at different scales.

**References**


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