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Fluid residence times within a recirculation zone created by an extraction–injection well pair

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Abstract

We present several analytical and semi-analytical solutions to evaluate the fluid residence times within the recirculation zone created by an extraction–injection well pair for several types of flow fields. The flow fields include: a well doublet in the absence of regional flow; an 'encaged recirculation cell' where a well doublet is located within and parallel to a uniform regional flow; and a well doublet with arbitrarily oriented uniform regional flow. For a well doublet in the absence of regional flow, we present an analytical solution for all streamlines. For the encaged recirculation cell, the first breakthrough time is solved analytically. We also develop a semi-analytical scheme to evaluate the residence time for any ratio of recirculated flow in a well doublet with arbitrarily oriented uniform regional flow. Both streamlines and travel times in the recirculation zone are strictly symmetric with respect to the midpoint between the two wells. We determine the starting points of streamlines at the well screen for a specific cumulative discharge by an analytical method. Using these starting points in a particle-tracking scheme, one is led directly to the cumulative breakthrough curve at the extraction well. Furthermore, we provide a convenient semi-analytical solution for the average residence time in the recirculation zone. The presented schemes minimize the number of particles that need to be tracked in the construction of breakthrough curves and help to efficiently design recirculation schemes for remediation purposes and to analyze tracer test data obtained in such systems.
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Keywords: Residence time; Analytical solution; Recirculation zone; Complex potential; Breakthrough curve

1. Introduction

Recirculation zones created by extraction–injection well pairs have been successfully used as in situ reactors to remediate contaminated groundwater (McCarty et al., 1998; Hyndman et al., 2000; Gandhi et al., 2002). These systems are effective in the mixing of dissolved compounds and the delivery of substrates. The fluid residence time within the reactor, i.e. the time to travel from the injection to the extraction well, is variable and, thus, the degree of treatment is generally variable. However, by increasing the number of times the flow circulates through the reactor, the total treatment time is increased and the variability is reduced through mixing within the wells (McCarty et al., 1998). Extraction–injection well pairs are also used in forced-gradient tracer tests to determine the hydraulic parameters (Grove and Beetem, 1971; Welty and Gelhar, 1994).

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In designing a recirculation zone, we need to consider its volume and residence time, and the rates of exchange between the reactor and the rest of the formation, i.e. the rate of inflow of contaminated water and the rate of outflow of treated water. Volume determination is related to the delineation of bounding streamlines, similar to capture zone delineation (Gorelick et al., 1993; Bakker and Strack, 1996; Shan, 1999; Christ and Goltz, 2002; Fienen et al., submitted). The recirculation ratio, i.e., the proportion of flow within the recirculation zone to the well flow rate, in homogeneous, isotropic aquifers can be derived analytically based on complex potential theory (Bear, 1979; Strack, 1989). Generally, two methods are used to evaluate the fluid residence time in the recirculation zone. The average residence time can be directly derived provided that the volume and the recirculation flow rate are known. The other commonly used method is streamline tracing: numerous particles are released at the injection-well boundary and move with the local seepage velocity until they reach the extraction well. The ensemble of all particle travel times yields the breakthrough curve at the extraction well (Zheng and Bennett, 2002).

Muskat (1937) determined the shape and position of a tracer front for a well-doublet flow field in the absence of regional flow and the first breakthrough time for the injected water reaching the extraction well. Bear (1979) studied the shape of the advancing front separating the indigenous water of a confined aquifer from a body of water injected into it by following the movement of water particles along streamlines. Zhan (1999a) defined a streamline constant, and derived closed-form steady-state analytical solutions for capture times for extraction–injection and extraction–extraction double-well systems in the absence of regional flow. Analytical and semi-analytical solutions of horizontal well capture times have also been studied by the same author (Zhan, 1999b; Zhan and Cao, 2000). In this paper, we derive an analytical solution using streamfunctions to determine the fluid residence times of a well doublet flow field in the absence of regional flow by taking advantage of the circular shape of streamlines. A semi-analytical scheme is also presented to evaluate the residence times for a well doublet flow field with arbitrarily oriented uniform regional flow.

2. Conceptual model

Consider a confined homogeneous and isotropic aquifer with uniform thickness b. Fig. 1 is a plan view of the flow field created by an extraction–injection well system with uniform regional flow (after McCarty et al., 1998). This system includes an extraction well, located at $z_E(d,0)$, and an injection well, located at $z_I(-d,0)$. The wells are located such that the midpoint between them is the origin of the coordinate system. The uniform regional flow is oriented at angle α from the positive x-axis. Water is pumped from the extraction well at a pumping rate Q_w and reinjected into the injection well at the same rate. Generally, the system has two stagnation points and the flow field can be divided into three zones of primary interest by the bounding streamlines passing through the stagnation points. Zone I is the capture zone, Zone II is the recirculation zone, and Zone III is the release zone. Outside of the bounding streamlines is regional flow not passing through any of the considered wells. We are interested in the recirculation zone only. However, under certain conditions

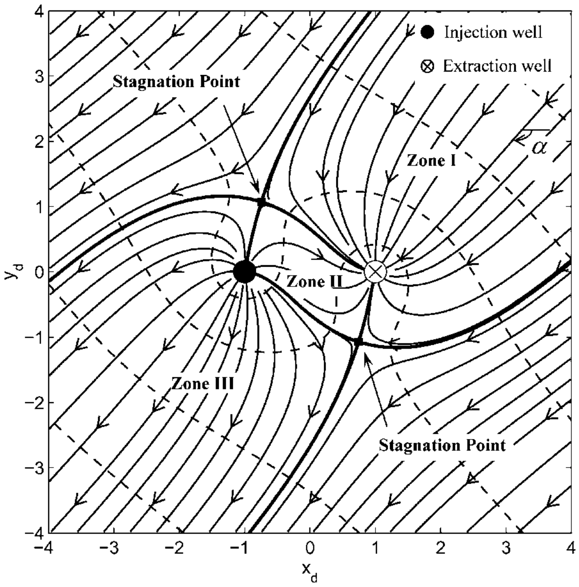

Fig. 1. Plan view of the flow field created by an extraction–injection well pair with uniform regional flow in a dimensionless domain. Solid lines are streamlines, dashed lines are hydraulic equipotential lines, and dark solid lines are separation streamlines.

one may observe that no streamline emerging from the injection well reaches the extraction well, i.e. no injected flow will ever reach the extraction well (Dacosta and Bennett, 1960; Bear, 1979; Erdmann, 2000).

3. Mathematical derivation

3.1. Complex functions

Complex potential theory is a convenient method for calculating two-dimensional groundwater flow in a homogeneous, isotropic medium assuming the Dupuit-Forcheimmer conditions. The notation and development presented follows Strack (1989).

$$\Omega(z) = \frac{Q_w}{2\pi}\ln\left(\frac{z - z_E}{z - z_I}\right) - \bar{Q}_0 z \tag{1}$$

$$\Omega(z) = \Phi(z) + \mathrm{i}\Psi(z), \qquad z = x + \mathrm{i}y, \qquad \bar{Q}_0 = Q_{x0} - \mathrm{i}Q_{y0} \tag{2}$$

where Ω is the complex potential, Φ is the discharge potential, Ψ is the streamfunction, Q_0 is the uniform discharge attributed to regional flow, $\bar{Q}_0$ indicates complex conjugate of Q_0. Q_{x0} and Q_{y0} are the components of regional flow rate in the x and y directions, respectively.

The complex discharge function is defined as

$$W(z) = -\frac{d\Omega}{dz} = \frac{-1}{2\pi}\frac{Q_w}{z - z_E} + \frac{1}{2\pi}\frac{Q_w}{z - z_I} + \bar{Q}_0 \tag{3}$$

$$W = Q_x - \mathrm{i}Q_y \tag{4}$$

and the seepage velocity can be calculated by

$$v = v_x + \mathrm{i}v_y = \frac{Q_x}{nb} + \mathrm{i}\frac{Q_y}{nb} = \frac{\bar{W}}{nb} \tag{5}$$

where b is the aquifer thickness and n is the effective porosity.

The discharge potential and streamfunction in the Cartesian coordinates are

$$\Phi = \frac{Q_w}{4\pi}\ln\left[\frac{(x-d)^2 + y^2}{(x+d)^2 + y^2}\right] - (xQ_{x0} + yQ_{y0}) \tag{6}$$

$$\Psi = \frac{Q_w}{2\pi}(\theta_1 - \theta_2) - (yQ_{x0} - xQ_{y0}) \tag{7}$$

where

$$\theta_1 = \arctan\left(\frac{y}{x-d}\right), \qquad \theta_1 \in [-\pi, \pi] \tag{8}$$

$$\theta_2 = \arctan\left(\frac{y}{x+d}\right), \qquad \theta_2 \in [-\pi, \pi] \tag{9}$$

The locations of the stagnation points can be calculated by solving a polynomial equation obtained by setting the discharge function $W = 0$.

$$z_s = \pm d\sqrt{1 + \frac{Q_w}{\pi\bar{Q}_0 d}} \tag{10}$$

where z_s indicates the location of stagnation point and d is the spatial distance of each well from the origin.

We introduce the following dimensionless variables. First, dimensionless spatial coordinates:

$$x_d = \frac{x}{d}, \qquad y_d = \frac{y}{d}, \qquad z_d = \frac{z}{d} = x_d + \mathrm{i}y_d \tag{11}$$

a dimensionless pumping rate λ :

$$\lambda = \frac{Q_w}{2\pi|Q_0|d} \tag{12}$$

The dimensionless discharge potentials and streamfunction are:

$$\Phi_d = \frac{\Phi}{|Q_0|d} = \frac{\lambda}{2}\ln\left[\frac{(x_d-1)^2 + y_d^2}{(x_d+1)^2 + y_d^2}\right] - (x_d\cos\alpha + y_d\sin\alpha) \tag{13}$$

$$\Psi_d = \frac{\Psi}{|Q_0|d} = \lambda(\theta_1 - \theta_2) - (y_d\cos\alpha - x_d\sin\alpha) \tag{14}$$

Finally, the dimensionless complex potential and discharge functions are:

$$\Omega_d = \frac{\Omega}{|Q_0|d} = \Phi_d + \mathrm{i}\Psi_d = \lambda\ln\left(\frac{z_d - 1}{z_d + 1}\right) - z_d\,e^{-\mathrm{i}\alpha} \tag{15}$$

$$W_\mathrm{d} = -\frac{\partial \Omega_\mathrm{d}}{\partial z_\mathrm{d}} = \frac{-2\lambda}{z_\mathrm{d}^2 - 1} + \mathrm{e}^{-\mathrm{i}\alpha} \tag{16}$$

The dimensionless position of the stagnation points are given as:

$$z_\mathrm{sd} = \pm\sqrt{1 + 2\lambda\,\mathrm{e}^{\mathrm{i}\alpha}} \tag{17}$$

Due to the discontinuity of the angle along the negative x-axis, the streamfunction of this system has a branch cut between the extraction and injection wells. A continuous streamfunction would require a flow field without internal volumetric sources or sinks.

According to the selected coordinate system, we can verify that:

- The discharge potential and streamfunction are odd functions with respect to the origin.

  $$\Phi(z) = -\Phi(-z) \tag{18}$$

  $$\Psi(z) = -\Psi(-z) \tag{19}$$

  Notice that the streamfunction remains an odd function despite the branch cut. Furthermore, Eq. (10) shows the two stagnation points, z_s1 and z_s2 are symmetric with respect to the origin. Thus, we have

  $$\Phi(z_\mathrm{s1}) = -\Phi(z_\mathrm{s2}) \tag{20}$$

  $$\Psi(z_\mathrm{s1}) = -\Psi(z_\mathrm{s2}) \tag{21}$$

- The discharge and velocity functions are even functions with respect to the origin.

  $$W(z) = W(-z) \tag{22}$$

  $$v(z) = v(-z) \tag{23}$$

- The recirculation zone has a symmetric shape with respect to the origin.

3.2. A well doublet in the absence of regional flow

Fig. 2 shows a doublet flownet in the vicinity of an extraction–injection well pair. The solid lines are streamlines and the dashed lines are equipotential lines. Notice that the flownet is a symmetric system, and equipotential lines and streamlines are circular.

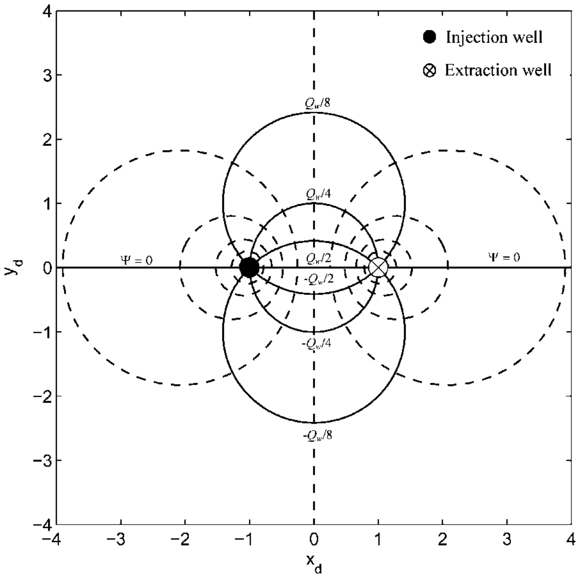

Fig. 2. A doublet flownet in the absence of regional flow.

The circle functions of equipotential lines are given by (Strack, 1989)

$$\left[x + d\coth\left(\frac{2\pi\Phi}{Q_\mathrm{w}}\right)\right]^2 + y^2 = \frac{d^2}{\sinh^2\left(\frac{2\pi\Phi}{Q_\mathrm{w}}\right)}, \quad \text{if } \Phi \neq 0 \tag{24}$$

$$x = 0, \quad \text{if } \Phi = 0 \tag{25}$$

and the circle functions of streamlines are given by (Strack, 1989)

$$x^2 + \left[y - d\cot\left(\frac{2\pi\Psi}{Q_\mathrm{w}}\right)\right]^2 = \frac{d^2}{\sin^2\left(\frac{2\pi\Psi}{Q_\mathrm{w}}\right)}$$

$$\text{if } \Psi \neq 0, \pm\frac{Q_\mathrm{w}}{2} \tag{26}$$

$$y = 0, x > d,\ x < -d, \qquad \text{if } \Psi = 0 \tag{27}$$

$$y = 0, d > x > -d, \qquad \text{if } \Psi = \pm\frac{Q_\mathrm{w}}{2} \tag{28}$$

Fig. 3 shows a streamline circle, which includes two streamlines starting from the injection well and ending at the extraction well. The difference of their streamfunction values is $Q_\mathrm{w}/2$. Table 1 lists the circle center, radius, length, height and central angle of

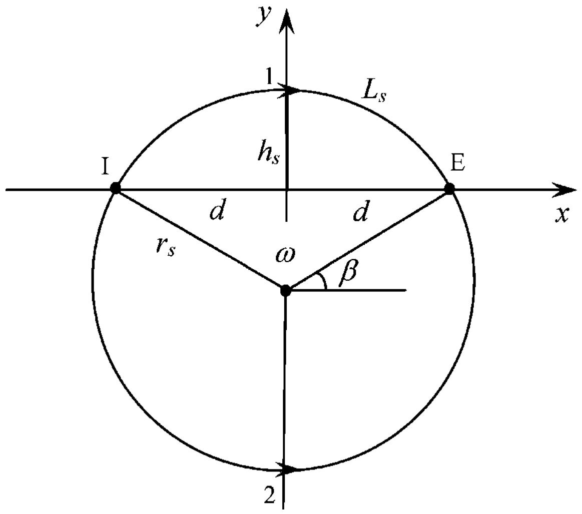

Fig. 3. Circular streamline and its properties.

the streamlines. Because flow moves along streamlines, it is more convenient to transform the Cartesian coordinates to polar coordinates with the origin located at the center of a given streamline circle.

$$x = x_c + r_s \cos(\beta) \tag{29}$$

$$y = y_c + r_s \sin(\beta) \tag{30}$$

Table 1
Properties of a streamline circle for a well doublet without regional flow

Streamline property	Value
Center	$(x_c, y_c) = \left(0, d\cot\left(\frac{2\pi\Psi}{Q_w}\right)\right)$
Radius	$r_s = \frac{d}{\left\|\sin\left(\frac{2\pi\Psi}{Q_w}\right)\right\|}$
Central angle	$\omega = 2\pi - 4\pi\frac{\|\Psi\|}{Q_w}$
Length of streamline	$L_s = r_s\omega = \frac{\omega d}{\left\|\sin\left(\frac{2\pi\Psi}{Q_w}\right)\right\|}$
Height of streamline arsc	$h_s = d\tan\left(\frac{\omega}{4}\right)$

where x_c and y_c are the Cartesian coordinates of the streamline circle center, r_s is the streamline circle radius, and β is the polar angle, $\beta \in [0, 2\pi]$.

Zhan (1999a) also transformed the Cartesian coordinates to polar coordinates, but with the polar origin located at (0, 0). Thus, the radius changes as a particle moves. Its velocity, in turn, must be evaluated both in radial and angular directions. In our coordinate system, r_s is constant along streamlines and we need to evaluate only the velocity component along the streamline, simplifying the calculation of velocities and travel times. Thus, the travel time can be expressed as

$$t = \int_{z_1}^{z_2} \frac{\mathrm{d}s}{|v|} = \int_{\beta_1}^{\beta_2} \frac{r_s|\mathrm{d}\beta|}{|v|} \tag{31}$$

where s is the length along the streamline, $|v|$ is the magnitude of seepage velocity.

Due to symmetry, it suffices to focus on the streamlines in the range of positive y values. As outlined in Appendix A1, the seepage velocity can be expressed as

$$|v| = \frac{Q_w d}{2\pi n b r_s}\frac{1}{r_s\sin(\beta) + y_c} = \frac{Q_w d}{2\pi n b r_s y} \tag{32}$$

The velocity magnitude is only a function of the y-coordinate for a selected streamline. The travel time from (x_1, y_1) to (x_2, y_2) along a streamline is given by

$$t = \int_{\beta_1}^{\beta_2} \frac{r_s|\mathrm{d}\beta|}{|v|} = \frac{2\pi n b r_s^2}{Q_w d}[y_c|\Delta\beta| + (x_2 - x_1)] \tag{33}$$

where $\Delta\beta$ is the streamline arc angle corresponding to the segment between (x_1, y_1) and (x_2, y_2).

On the other hand, provided with an initial point and a travel time, Eq. (33) can be transformed to an implicit equation to calculate the ending point. Thus, if the starting point is the injection well, Eq. (33) can determine the shape and position of a tracer front, which is the solution provided in Muskat (1937); if the ending point is the extraction well, Eq. (33) can calculate the capture time for any starting point, which is the solution provided in Zhan (1999a). Eq. (33) is simpler and more versatile.

Substituting the values listed in Table 1, the arrival/breakthrough time, from the injection well to the extraction well, can be derived as

$$t = \frac{2\pi nb(y_c\omega + 2d)r_s^2}{Q_w d} = \frac{4\pi nb}{\sin^2\left(\frac{2\pi|\Psi|}{Q_w}\right)} \times \left[1 + \pi\left(1 - 2\frac{|\Psi|}{Q_w}\right)\cot\left(2\pi\frac{|\Psi|}{Q_w}\right)\right]\frac{d^2}{Q_w} \quad (34)$$

which in dimensionless form is

$$\tau = \frac{t}{T} = \frac{1 + \pi(1 - 2|\Psi_d|)\cot(2\pi|\Psi_d|)}{\sin^2(2\pi|\Psi_d|)} \quad (35)$$

where T is a characteristic time of the well system defined by

$$T = \frac{4\pi nbd^2}{Q_w} \quad (36)$$

and Ψ_d is the dimensionless streamfunction

$$\Psi_d = \frac{\Psi}{Q_w} \quad (37)$$

The first breakthrough t_0 is obtained at the shortest streamline connecting the injection well to the extraction well, having the streamfunction value $|\Psi| = Q_w/2$ (see Appendix A2).

$$t_0 = \frac{4\pi nb}{3}\frac{d^2}{Q_w} \quad (38)$$

The dimensionless first breakthrough time is $\tau_0 = 1/3$, which can also be derived from Eq. (35) when $|\Psi_d|$ approaches to 1/2 (see Appendix A2). The median dimensionless breakthrough time, when half the flow reaches the extraction well, is $\tau_{50} = 1$ given by $|\Psi| = Q_w/4$.

Fig. 4 shows the dimensionless arrival time versus the dimensionless streamfunction. τ decreases with the increase of the absolute value of Ψ_d. When Ψ_d approaches to zero, flow leaves the injection well along the negative x direction and takes a theoretically infinite time to reach the extraction well. Fig. 5 shows the cumulative breakthrough curve at the extraction well, which is directly derived from Fig. 4 and Eq. (35). Notice that Eq. (35) shows dimensionless arrival times are only related to dimensionless streamfunction, not λ. Thus, for a streamline with a given dimensionless streamfunction its dimensionless arrival time is constant for any Q_w and d, but the dimensional arrival time is proportional to d^2/Q_w for a given n and b.

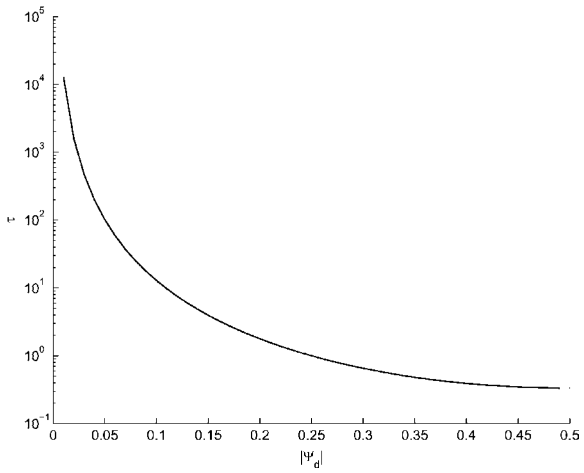

Fig. 4. Dimensionless arrival time versus dimensionless streamfunction for the well doublet system without regional flow.

3.3. A well doublet with arbitrary uniform regional flow

Including uniform regional flow to the well-doublet flow field, the shape of streamlines becomes less regular (Fig. 1), and the velocity function cannot be transformed to a formula as simple as Eq. (32). Generally, simple analytical solutions can be derived only under specific conditions. We assume the well radius is r_w, and choose two particles symmetrically

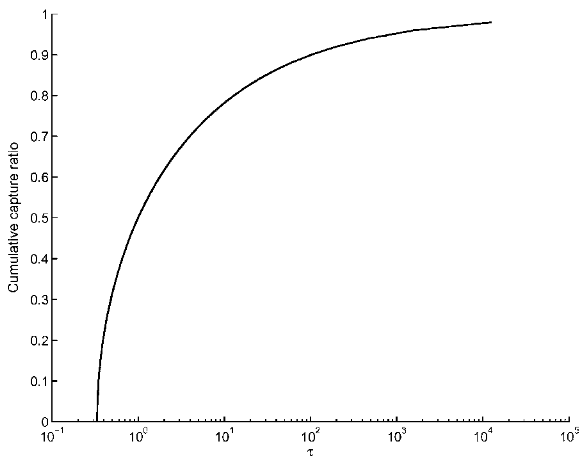

Fig. 5. Cumulative breakthrough curve at the extraction well in a well doublet in the absence of regional flow.

located with respect to the origin. One particle is located at the injection-well screen, z_1^0, and the other particle is located at the extraction well screen, z_2^0. Due to symmetry, we have:

$$z_1^0 = -z_2^0 \tag{39}$$

where the subscript represents the particle number and the superscript represents the tracing step.

Thus:

$$\Psi(z_1^0) = -\Psi(z_2^0), \qquad v_x(z_1^0) = v_x(z_2^0), \qquad v_y(z_1^0) = v_y(z_2^0) \tag{40}$$

The streamline starting from z_1^0 leaves the injection well and moves to the extraction well or leaves the system with the regional flow. We use a forward-tracking scheme for z_1^0 and a backward-tracking scheme for z_2^0 to trace streamlines.

$$z_1^1 = z_1^0 + \int_0^{\Delta t} v_1\, \mathrm{d}t, \qquad z_1^i = z_1^{i-1} + \int_0^{\Delta t} v_1\, \mathrm{d}t \tag{41}$$

$$z_2^1 = z_2^0 - \int_0^{\Delta t} v_2\, \mathrm{d}t, \qquad z_2^i = z_2^{i-1} - \int_0^{\Delta t} v_2\, \mathrm{d}t \tag{42}$$

where z_1^i and z_2^i are the particle locations on streamlines 1 and 2 after the ith tracing step, respectively, and v_1 and v_2 are the corresponding local seepage velocities.

According to the relations between z_1^0 and z_2^0, we have:

$$z_2^1 = -z_1^0 - \int_0^{\Delta t} v_1\, \mathrm{d}t = -z_1^1 \tag{43}$$

and

$$z_2^i = -z_1^{i-1} - \int_0^{\Delta t} v_1\, \mathrm{d}t = -z_1^i \tag{44}$$

Thus, we can reach two conclusions: (1) streamlines 1 and 2 are symmetric with respect to the origin; (2) streamlines 1 and 2 have the same travel times.

Consider a streamtube with two symmetric bounding streamlines in the recirculation zone. The flow discharge within this streamtube equals the difference in value of streamfunctions on these streamlines. On the other hand, streamfunction within this streamtube is discontinuous due to the branch cut between the two wells. Since the streamfunction jumps by an amount of Q_w across the cut, the discharge inside this streamtube is

$$\Delta Q_\mathrm{d} = 2\pi\lambda - |\Psi_\mathrm{d}(z_1^0) - \Psi_\mathrm{d}(z_2^0)| = 2\pi\lambda - 2|\Psi_\mathrm{d}(z_1^0)| \tag{45}$$

The first breakthrough is given under the condition

$$\Delta Q_\mathrm{d} = 0 \tag{46}$$

Thus, we have

$$|\Psi_\mathrm{d}(z_1^0)| = \pi\lambda = |\Psi_\mathrm{d}(z = 0)| \tag{47}$$

Generally, the streamline with the first breakthrough time is assumed to be a straight line connecting the two wells. However, this assumption is valid only when there is no regional flow or the regional-flow direction is parallel to the two wells. In general, the streamline with the first breakthrough is not a straight line connecting the two wells (see Fig. 6). In fact, the straight segment connecting the two wells is not even a streamline when the regional flow is not parallel to the well placement. However, Eq. (47) shows that the streamline with the first breakthrough time always passes through the midpoint between the two wells. For this streamline, the forward tracking from the injection-well screen to the extraction-well screen is identical to the backward tracking starting from the symmetric point at the extraction-well screen. Based

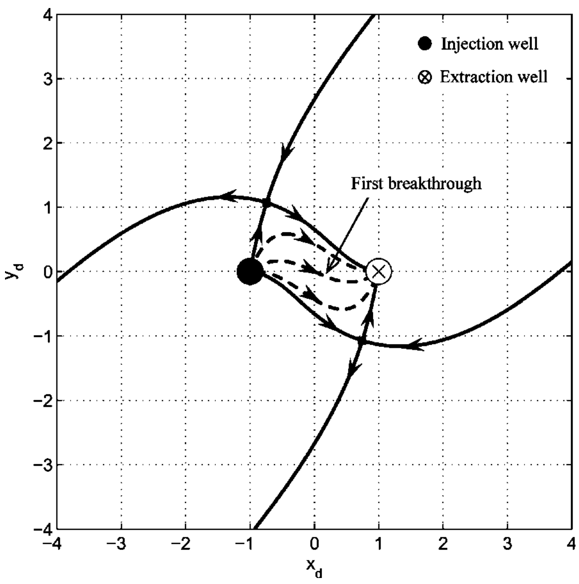

Fig. 6. Symmetric streamlines and travel times. The streamline with the first breakthrough time passes through the origin.

on this analysis, we can easily approximate the first breakthrough: insert a particle at the origin and track it to the extraction well. The time for the particle to reach the extraction well is one-half of the first breakthrough time.

If the starting points have the same streamfunctions as the stagnation points, the streamtube between these two streamlines comprises the recirculation zone. Thus, the total flow rate in the recirculation zone in an extraction–injection well system is given by:

$$Q_r = Q_w - 2|\Psi_s| \tag{48}$$

$$Q_{rd} = \frac{Q_r}{|Q_0|d} = 2\pi\lambda - 2|\Psi_{sd}| \tag{49}$$

where Q_r and Q_{rd} are the discharge in the recirculation zone and its dimensionless formula, respectively; Ψ_s is the streamfunction of a stagnation point; and $\Psi_{sd} = (\Psi_s)/|Q_0 d|$, is the dimensionless streamfunction at the stagnation point.

The total recirculation ratio in the recirculation zone is given by

$$P_r = \frac{Q_r}{Q_w} = 1 - \frac{|\Psi_{sd}|}{\pi\lambda} \tag{50}$$

If $P_r = 0$, i.e. $|\Psi_{sd}| = \pi\lambda$, a minimum λ can be obtained by solving this implicit equation. This is the critical condition to have recirculation in the extraction–injection well system; that is, when λ is greater than its 'critical' value, there is flow between the injection and extraction wells. In this paper, we assume this critical condition is satisfied, i.e. there is a recirculation zone between the injection and extraction wells.

Travel times increase with the cumulative capture ratio, P, so travel times of the streamlines between the streamline with the first breakthrough and the streamline through a stagnation point are monotonic, i.e. the closer to the origin, the smaller the travel time. Eq. (45) provides us with a fast tool to find the streamlines with a specific residence time. For example, if we want to obtain the median residence time in the recirculation zone, we have

$$P = \frac{1}{2} = \frac{\Delta Q_d}{Q_{rd}} = \frac{2\pi\lambda - 2|\Psi_d(z_0)|}{2\pi\lambda - 2|\Psi_{sd}|} \tag{51}$$

$$|\Psi_d(z_0)| = \frac{\pi\lambda}{2} + \frac{|\Psi_{sd}|}{2} \tag{52}$$

where z_0 is the starting point located at the injection-well screen, and can be expressed as

$$z_0 = z_I + r_w\, e^{i\gamma} \tag{53}$$

where γ is the phase angle.

Thus, we may trace just this streamline to obtain the median residence time. Generally, the breakthrough time for a specific P, in the recirculation zone, can be determined by selecting the streamline with the corresponding streamfunction value:

$$|\Psi_d(z_0)| = (1 - P)\pi\lambda + P|\Psi_{sd}| \tag{54}$$

Substituting Eqs. (53) and (54) into Eq. (14), γ can be obtained by solving

$$\begin{aligned}\lambda\left[\tan^{-1}\left(\frac{\sin(\gamma)}{\cos(\gamma)+1}\right) - \gamma\right] &- [r_d \sin(\gamma)\cos(\alpha) \\ &+ (1 - r_d\cos(\gamma))\sin(\alpha)] \\ &= (1-P)\pi\lambda + P|\Psi_{sd}|\end{aligned} \tag{55}$$

This semi-analytical algorithm can be summarized as follows:

1. Calculate the locations of stagnation points using Eq. (17).
2. Calculate the values of the streamfunction at the stagnation points using Eq. (14).
3. Calculate the values of the streamfunctions corresponding to capture ratios using Eq. (54).
4. Locate the starting points at the injection-well screen by solving Eqs. (53) and (55).
5. Trace streamlines to evaluate the breakthrough times.

3.4. Average residence time

The recirculation zone may be modeled as a plug-flow reactor. Consider an ideal plug-flow reactor (i.e. subject only to advection) with constant empty-bed volume V, subject to discharge Q. The simplest case, of course, is that of uniform specific discharge $q = Q/A$, where A is the uniform cross-sectional area. Then, the residence time is

$$t = \frac{nAL}{Q} = \frac{nV}{Q} \tag{56}$$

where L is the length of the reactor. However, if the velocity within the recirculation zone is spatially variable, the zone can be modeled as a system of streamtubes with different residence times in each streamtube. Consider steady-state flow in a very narrow streamtube with discharge dQ.

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}Q}{n\,\mathrm{d}A} \tag{57}$$

or

$$\mathrm{d}t = \frac{n\,\mathrm{d}A\,\mathrm{d}s}{\mathrm{d}Q} \tag{58}$$

Since dQ is constant, the total time to travel distance L in an individual streamtube is

$$t_\mathrm{s} = \frac{n\,\mathrm{d}V}{\mathrm{d}Q} \tag{59}$$

where dV is the streamtube volume. The residence time weighted-averaged with the discharge over all streamtubes is

$$t_\mathrm{a} = \frac{\int t_\mathrm{s}\,\mathrm{d}Q}{\int \mathrm{d}Q} = \frac{nV}{Q} \tag{60}$$

Thus, the average residence time is the reactor volume divided by the discharge. This simple relationship applies to any reactor type. It is the residence time of the water but also the residence time of a solute that undergoes only advection.

The total flow rate in the recirculation zone in an extraction–injection well system is already given by Eq. (48). And the area of the recirculation zone can be evaluated using Green's theorem in the plane.

$$A = \frac{1}{2}\oint_C (-y\,\mathrm{d}x + x\,\mathrm{d}y) \tag{61}$$

A numerical approximation given by n points along the bounding streamlines is:

$$A = \frac{1}{2}\sum_{k=1}^{n} (-Y_k\,\mathrm{d}X_k + X_k\,\mathrm{d}Y_k) \tag{62}$$

where k indicates the segment between vertices k and $k+1$; X_k and Y_k are the coordinates of the center of segment k; and $\mathrm{d}X_k = x_{k+1} - x_k$ and $\mathrm{d}Y_k = y_{k+1} - y_k$.

These points are located at the bounding streamlines passing through the stagnation points, and can be obtained by tracing streamlines from the stagnation points or by other methods for capture zone delineation (Fienen et al., submitted). Thus, the average residence time can be evaluated by

$$t_\mathrm{a} = \frac{nAb}{Q_\mathrm{r}} \tag{63}$$

4. Applications

4.1. Encaged recirculation cell

Consider a scenario where the extraction well is located downgradient from the injection well and the uniform regional flow is parallel to the well placement. The recirculation cell becomes completely encaged and isolated from the outside regional flow (Fig. 7). The two stagnation points are on the x-axis, the values of their streamfunctions are zero, and are located at

$$z_\mathrm{sd} = \pm\sqrt{1+2\lambda} \tag{64}$$

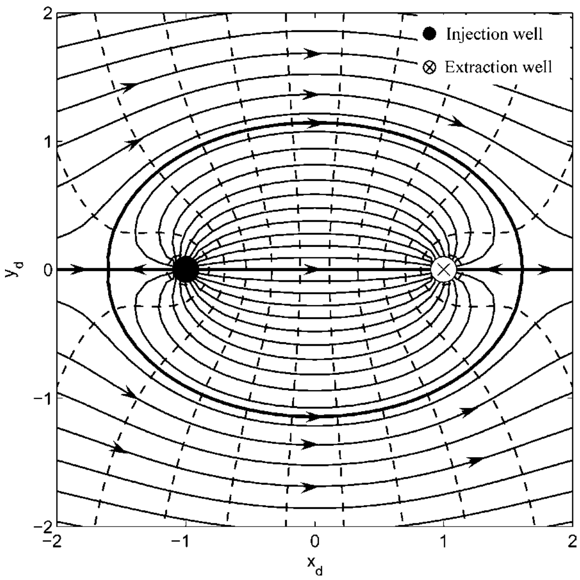

Fig. 7. An 'encaged recirculation cell' created by an extraction–injection well pair. The injection well is located upgradient and the uniform regional flow is parallel to the axis connecting the wells.

The first breakthrough time can be derived analytically (see Appendix A3).

$$t_0 = \frac{2d}{v_{0x}} - \frac{2dv_{wx}}{v_{0x}^2}\sqrt{\frac{v_{0x}}{v_{0x}+v_{wx}}}\tanh^{-1}\left(\sqrt{\frac{v_{0x}}{v_{0x}+v_{wx}}}\right) \tag{65}$$

where

$$v_{0x} = \frac{Q_{0x}}{nb} = \frac{Q_0}{nb} \tag{66}$$

$$v_{wx} = \frac{Q_\mathrm{w}}{\pi nbd} \tag{67}$$

The dimensionless expression is

$$\tau_0 = \frac{t_0}{T^*} = 1 - 2\lambda\sqrt{\frac{1}{1+2\lambda}}\tanh^{-1}\sqrt{\frac{1}{1+2\lambda}} \tag{68}$$

where

$$T^* = \frac{2d}{v_{0x}} \tag{69}$$

Fig. 8 shows that τ_0 declines with the increase of λ. When λ approaches to zero, τ_0 is controlled by regional flow only. When $\lambda \gg 1$, the regional flow can be neglected and τ_0 is controlled by the wells only. In this case, the dimensional breakthrough time approaches to Eq. (38), as shown in Appendix A3.

To evaluate the average residence time within the encaged cell, an explicit solution can be obtained to specify the bounding streamlines and to calculate the area of the recirculation zone.

$$x_\mathrm{d}^2 = 1 - y_\mathrm{d}^2 + 2y_\mathrm{d}\cot\left(\frac{y_\mathrm{d}}{\lambda}\right), \qquad y_\mathrm{d} \neq 0 \tag{70}$$

Integration yields:

$$A_\mathrm{d} = 2\pi\lambda \tag{71}$$

and thus in dimensional form:

$$A = A_\mathrm{d}d^2 = \frac{Q_\mathrm{w}d}{Q_0} \tag{72}$$

Therefore, the average residence time is

$$t_\mathrm{a} = \frac{nA_\mathrm{d}d^2b}{Q_\mathrm{w}} = \frac{n\,db}{Q_0} \tag{73}$$

From Eq. (73) we see the average residence time within the closed loop is proportional to the well spacing and, somewhat surprisingly, does not depend on the pumping rate. This is a direct consequence of the fact that the cell volume is proportional to the pumping rate. For constant regional flow, the only way to change the average residence time is to adjust the well spacing. Generally, two factors control the breakthrough time of a streamline: the streamline length and the velocity. Increasing the pumping increases both the length of the streamlines and the velocity. For the inner streamlines (closer to the x-axis), such as the streamline with the first breakthrough time, the effect of adjusting the pumping rates on velocity is dominant. Therefore, increasing the pumping rates results in a consistent decrease of the first breakthrough times as shown in Fig. 8. For the outer streamlines, however, the effect of elongating the streamlines is more significant. Thus, their breakthrough times behave conversely. Fig. 9 shows that τ_{90} consistently increases within the range of $\lambda \leq 1000$. τ_{90} represents the residence time in which 90% of the recirculation flow is captured by the extraction well. The effects of inner streamlines and outer streamlines counteract so that the overall average residence time remains constant.

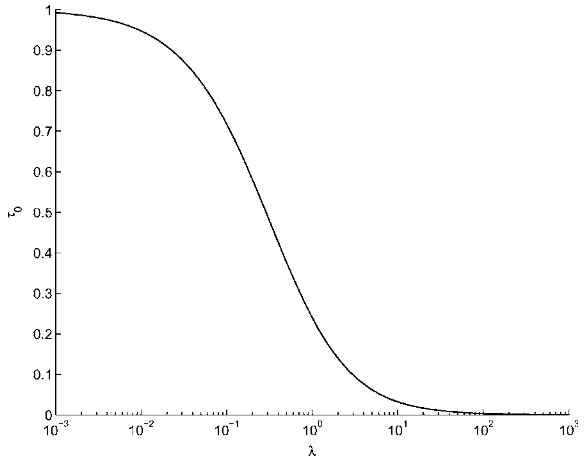

Fig. 8. The dimensionless first breakthrough time versus λ within the 'encaged recirculation cell'.

4.2. Arbitrary regional flow

In this case, the uniform regional flow is assumed to be 45° with the vector from the injection well to the extraction well A conservative

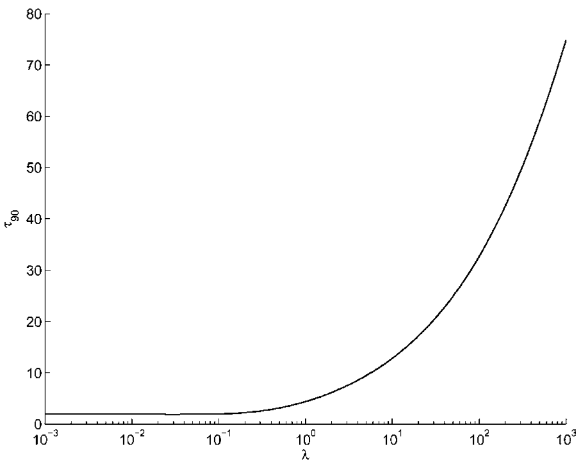

Fig. 9. The dimensionless breakthrough time of 90% recirculated flow in the 'encaged recirculation cell' versus λ.

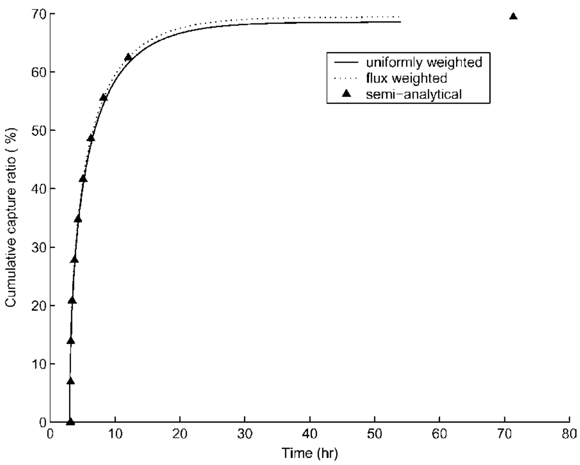

Fig. 10. Cumulative breakthrough curve at the extraction well for the well doublet flow field with 45° uniform regional flow.

tracer is released as a step function into the injection well. Consider only advection. Two methods are used to evaluate the cumulative breakthrough curve at the extraction well. The first one is particle tracking with 3600 particles starting uniformly distributed at the injection-well screen. The second one uses the scheme we presented here to specify the starting points with the first breakthrough time and breakthrough times of 10,20,…,100% recirculation flow. The uniform regional flow rate and well flow rate are 10^{-4} and 10^{-3} m^2/s, respectively. Aquifer thickness is 10 m. Porosity is 0.3. The well radius and spacing are 0.05 and 2 m, respectively. The total recirculation ratio is 69.4%, which can be obtained analytically by solving Eq. (50).

Fig. 10 compares the results obtained by these two schemes. The accuracy of the conventional scheme depends on the number of particles and the spatial as well as temporal resolutions. Particles are assumed uniformly weighted or flux weighted. The first approach is simpler, but the second one is more accurate. The results plotted in Fig. 10 show minor discrepancies between the two approaches. Using the semi-analytical scheme, we determined first the starting locations for specified values of the cumulative capture ratio and subsequently calculated the breakthrough times. Because we use fewer particles, we can use a finer spatial and temporal discretization in the evaluation of the breakthrough times so that the results are more accurate. Especially when a special residence time is selected as a constraint in design, such as the median residence time, this scheme can reduce the computational costs significantly.

5. Summary and conclusions

First, we derived an analytical solution to calculate the fluid residence time in the recirculation zone created by an extraction–injection well pair in the absence of regional flow. For a selected streamline, the magnitude of seepage velocity is found to be a function of only the y coordinate (the two wells located on the x-axis). The dimensionless residence times are constants and the dimensional residence times are proportional to d^2/Q_w, where d is the half distance between the two wells, and Q_w is the pumping rate.

Second, for the encaged recirculation cell, in which the extraction well is placed downgradient from the injection well, an analytical solution for the first breakthrough time was derived. The average residence time within the cell is not affected by the value of Q_w. The reason is that as Q_w increases, the travel time in the inner streamlines decreases but the size of the cell increases and streamlines of low velocity are created.

Finally, an efficient semi-analytical solution algorithm was developed to evaluate the fluid residence time within the recirculation zone created by a well doublet with arbitrarily oriented uniform regional flow. Streamlines and arrival times are strictly symmetric with respect to the midpoint between the two wells. Particularly, the streamline with the first breakthrough time always passes through the midpoint between the wells. Thus, we present a method to trace the streamline starting from the origin to evaluate the first breakthrough time. The semi-analytical solution for the well system with arbitrarily uniform regional flow allows us to determine the starting points of streamlines at the injection-well screen for a specified cumulative capture ratio which can be exploited to calculate cumulative breakthrough curves. By minimizing the number of particles to be tracked, our solutions reduce the computational effort and provide an efficient tool to help design remediation schemes based on an extraction–injection well pair and analyzing tracer tests implemented in such systems.

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Appendix A

A.1. Velocity in a well doublet in the absence of regional flow

Based on Eqs. (3) and (5), the magnitude of seepage velocity in the absence of regional flow can be calculated as

$$|v| = \frac{|W|}{nb} = \frac{Q_w d}{\pi nb}[(x^2 - y^2 - d^2)^2 + 4x^2y^2]^{-1/2}$$
$$= \frac{Q_w d}{\pi nb}[(x^2 + y^2 + d^2)^2 - 4x^2d^2]^{-1/2}$$

By replacing x and y with Eqs. (29) and (30), we have

$$\begin{aligned}|v| &= \frac{Q_w d}{\pi nb}\left[\left(r_s^2 + d^2\cot^2\left(\frac{2\pi\Psi}{Q_w}\right)\right.\right.\\ &\qquad \left.+ 2\,\mathrm{d}r_s\cot\left(\frac{2\pi\Psi}{Q_w}\right)\sin(\beta) + d^2\right)^2\\ &\qquad \left.-4r_s^2d^2\cos^2(\beta)\right]^{-1/2}\\ &= \frac{Q_w d}{2\pi nb}\left[\left(r_s^2 + \mathrm{d}r_s\cot\left(\frac{2\pi\Psi}{Q_w}\right)\sin(\beta)\right)^2\right.\\ &\qquad \left.- r_s^2d^2\cos^2(\beta)\right]^{-1/2}\end{aligned}$$

Let

$$f_1 = \frac{Q_w d}{2\pi nb}$$

$$f_2 = r_s^2$$

$$f_3 = \mathrm{d}r_s\cot\left(\frac{2\pi\Psi}{Q_w}\right)$$

$$f_4 = r_s^2d^2$$

Then,

$$|v| = f_1[(f_2 + f_3\sin(\beta))^2 - f_4\cos^2(\beta)]^{-1/2}$$
$$= f_1[f_2^2 + 2f_2f_3\sin(\beta) - f_4 + (f_3^2 + f_4)\sin^2(\beta)]^{-1/2}$$

Let

$$g_1 = f_2^2 - f_4 = r_s^2(r_s^2 - d^2) = r_s^2y_c^2$$

$$g_2 = 2f_2f_3 = 2r_s^3y_c$$

$$g_3 = f_3^2 + f_4 = y_c^2r_s^2 + r_s^2d^2 = r_s^4$$

Then,

$$|v| = f_1[r_s^2(y_c^2 + 2r_sy_c\sin(\beta) + r_s^2\sin^2(\beta))]^{-1/2}$$
$$= \frac{Q_w d}{2\pi nbr_s}\frac{1}{|r_s\sin(\beta) + y_c|}$$

A.2. The first breakthrough for a well doublet in the absence of regional flow

For the direct streamline from injection well to extraction well, we have

$$y = 0, d > x > -d$$

$$\Psi = \pm \frac{Q_w}{2}$$

and

$$v = \frac{Q_w d}{\pi nb} \frac{1}{(d^2 - x^2)}$$

Thus,

$$t_0 = \int_{-d}^{d} \frac{dx}{v} = \frac{4\pi nb}{3} \frac{d^2}{Q_w}$$

In dimensionless form, $\tau_0 = 1/3$, which can also be derived from Eq. (35) when $|\Psi_d|$ approaches to 1/2.

$$\tau_0 \overset{|\Psi_d| \to 1/2}{=} \frac{1 + \pi(1 - 2|\Psi_d|)\cot(2\pi|\Psi_d|)}{\sin^2(2\pi\Psi_d)} = \frac{\sin(2\pi|\Psi_d|) + \pi(1 - 2|\Psi_d|)\cot(2\pi|\Psi_d|)}{\sin^3(2\pi\Psi_d)}$$

$$\approx \frac{-(2\pi|\Psi_d| - \pi) + (2\pi|\Psi_d| - \pi)^3/6 - (\pi - 2\pi|\Psi_d|) - (2\pi|\Psi_d| - \pi)^3/2}{(\pi - 2\pi|\Psi_d|)^3} = \frac{1}{3}$$

A.3. The first breakthrough for an encaged recirculation cell

The streamline with the first breakthrough time is the direct path connecting the two wells. Thus, its velocity is parallel to the x-axis.

$$v_x = \frac{dx}{dt} = \frac{Q_w}{2\pi nb}\left(\frac{1}{x+d} - \frac{1}{x-d}\right) + \frac{Q_{0x}}{nb}$$

$$= \frac{Q_{0x}}{nb} - \frac{Q_w}{\pi nb} \frac{d}{x^2 - d^2}$$

Let

$$\begin{cases} v_{0x} = \dfrac{Q_{0x}}{nb} \\ v_{wx} = \dfrac{Q_w}{\pi nbd} \end{cases}$$

Then,

$$dt = \frac{dx}{v_{0x} - \dfrac{v_{wx}d^2}{x^2 - d^2}} = \frac{(x^2 - d^2)dx}{v_{0x}(x^2 - d^2) - v_{wx}d^2}$$

and

$$t = \int_{-d}^{d} \frac{(x^2 - d^2)}{v_{0x}(x^2 - d^2) - v_{wx}d^2} dx$$

$$= \frac{2d}{v_{0x}} - \frac{2\,dv_{wx}}{v_{0x}^2}\sqrt{\frac{v_{0x}}{v_{0x} + v_{wx}}}\tanh^{-1}\left(\sqrt{\frac{v_{0x}}{v_{0x} + v_{wx}}}\right)$$

When the uniform regional flow rate is much smaller than the well flow rate, $v_{0x} \ll v_{wx}$, we have:

$$t \approx \frac{2d}{v_{0x}} - \frac{2dv_{wx}}{v_{0x}^2} \times \sqrt{\frac{v_{0x}}{v_{0x} + v_{wx}}}\left[\sqrt{\frac{v_{0x}}{v_{0x} + v_{wx}}} + \frac{1}{3}\left(\frac{v_{0x}}{v_{0x} + v_{wx}}\right)^{3/2}\right]$$

$$= \frac{2d}{v_{0x}} - \frac{2\,dv_{wx}}{v_{0x}^2}\left[\frac{v_{0x}}{v_{0x} + v_{wx}} + \frac{1}{3}\left(\frac{v_{0x}}{v_{0x} + v_{wx}}\right)^2\right]$$

$$= \frac{6\,dv_{0x} + 4\,dv_{wx}}{3(v_{0x} + v_{wx})^2} \approx \frac{4d}{3v_{wx}} = \frac{4\pi nb}{3} \frac{d^2}{Q_w}$$

Thus, when the regional flow rate is much smaller, the solution approaches to the one obtained in the absence of regional flow.

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