Appendix D

The Generalized (Power Transformed) F Family of Nonnegative Probability Distributions

Probability Distribution	Probability Density Function (PDF)	Cumulative Distribution Function (CDF)	*r*th moment E(Xr)
General X with parameter vector 2	$f_X(x \mid \theta) = F_X'(x \mid \theta)$ or $f_X(x) = F_X'(x)$(2 suppressed)	$F_X(x \mid \theta) = Prob\,[X \le x \mid \theta]$	$\mu_X(r, \theta) = \int_0^\infty x^r f_X(x \mid \theta)\,dx$
Monotonic transformation $Y = g(X),\ X = h(Y) = g^{-1}(Y)$	$f_Y(y \mid \theta) = f_X[h(y) \mid \theta]\,\lvert h'(y) \rvert$	$F_Y(y \mid \theta) = F_X[h(y) \mid \theta]$ for h increasing $F_Y(y \mid \theta) = 1 - F_X[h(y) \mid \theta]$ for h decreasing	$\int_{g(0)}^{g(\infty)} y^r f_Y(y)\,dy$ $= \int_0^\infty g(x)^r f_X(x)\,dx$
$Y = 1/X$	$f_Y(y) = f_X(1/y)/y^2$	$F_Y(y) = 1 - F_X(1/y)$	$\mu_Y(r, \alpha_1, \alpha_2, \lambda, \sigma)$ $= \mu_X(r, \alpha_1, \alpha_2, \lambda, -\sigma)$ $= \mu_X(r, \alpha_2, \alpha_1, \lambda, \sigma)$
Generalized F (GF4) $\theta = (\alpha_1, \alpha_2, \lambda, \sigma) > 0$ $p = 1/\sigma, \lambda = \exp(-\mu)$	$\dfrac{p\,[\alpha_1(\lambda x)^p/\alpha_2]^{\alpha_1}}{B(\alpha_1,\alpha_2)\,x\,[1+\alpha_1(\lambda x)^p/\alpha_2]^{\alpha_1+\alpha_2}}$	$PROBF\left((\lambda x)^p, 2\alpha_1, 2\alpha_2\right)$	$\left(\dfrac{\alpha_2}{\alpha_1}\right)^{r\sigma} \dfrac{\Gamma(\alpha_1 + r\sigma)\,\Gamma(\alpha_2 - r\sigma)}{\lambda^r\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)}$
Generalized gamma (GG3) $\theta = (\alpha, \beta, p) > 0$ GF: $\alpha_1 = \alpha,\ \alpha_2 \to \infty,\ \beta = \alpha^{-\sigma} e^{\mu},\ p = 1/\sigma$	$\dfrac{p\,x^{\alpha p - 1}\,\exp[-(x/\beta)^p]}{\Gamma(\alpha)\,\beta^{\alpha p}}$	$PROBGAM\left((x/\beta)^p, \alpha\right)$	$\dfrac{\Gamma(\alpha + r\sigma)}{(\lambda\,\alpha^\sigma)^r\,\Gamma(\alpha)} = \dfrac{\beta^r\,\Gamma(\alpha + r\sigma)}{\Gamma(\alpha)}$
Burr/Dubey (Bur3) $\theta = (\alpha, \lambda, p) > 0$ GF: $\alpha_1 = 1,\ \alpha_2 = \alpha$	$\dfrac{\lambda\,p\,(\lambda x)^{p-1}}{[1 + (\lambda x)^p/\alpha]^{1+\alpha}}$	$1 - \dfrac{1}{[1 + (\lambda x)^p/\alpha]^{\alpha}}$	$\dfrac{\alpha^{r\sigma}\,\Gamma(1 + r\sigma)\,\Gamma(\alpha - r\sigma)}{\lambda^r\,\Gamma(\alpha)}$
Gumbel generalized logistic (Gum3) $\theta = (\alpha, \lambda, p) > 0$ GF: $\alpha_1 = \alpha_2 = \alpha$	$\dfrac{\lambda\,p\,(\lambda x)^{\alpha p - 1}}{B(\alpha, \alpha)\,[1 + (\lambda x)^p]^{2\alpha}}$	$PROBF\left((\lambda x)^p, 2\alpha, 2\alpha\right)$	$\dfrac{\Gamma(\alpha + r\sigma)\,\Gamma(\alpha - r\sigma)}{\lambda^r\,[\Gamma(\alpha)]^2}$

Probability Distribution	Probability Density Function (PDF)	Cumulative Distribution Function (CDF)	*r*th moment E(Xr)
Gamma (Gam2) $\theta = (\alpha, \beta) > 0$ GF: $\alpha_1 = \alpha,\ \alpha_2 \to \infty,\ \sigma = p = 1,\ \beta = e^{\mu}$	$\dfrac{x^{\alpha-1}\exp(-x/\beta)}{\Gamma(\alpha)\beta^{\alpha}}$	$PROBGAM(x/\beta, \alpha)$	$\dfrac{\beta^r\,\Gamma(\alpha+r)}{\Gamma(\alpha)}$
Lognormal (Log2) $\theta = (\mu, \sigma),\ \sigma > 0,\ \mu$ *real* GF: $\alpha_1 \to \infty,\ \alpha_2 \to \infty$	$\dfrac{\exp\{-[(\log(x)-\mu)^2]/[2\sigma^2]\}}{\sqrt{2\pi}\ \sigma x}$	$PROBNORM\{[\log(x)-\mu]/\sigma\}$	*exp (r : +F²r².2)*
Weibull (Wei2) $\theta = (\lambda, p) > 0$ GF: $\alpha_1 = 1,\ \alpha_2 \to \infty$	$\lambda p(\lambda x)^{p-1}\ \exp[-(\lambda x)^p]$	$1 - \exp[-(\lambda x)^p]$	$\dfrac{\Gamma(1+r/p)}{\lambda^r}$
Log-logistic (Tic2) $\theta = (\lambda, p) > 0$ GF: $\alpha_1 = \alpha_2 = 1$	$\lambda\ p\ (\lambda x)^{p-1}/[1+(\lambda x)^p]^2$	$(\lambda x)^p / [1+(\lambda x)^p]\, F_Y(y)$	$\dfrac{\Gamma(1+r\sigma)\ \Gamma(1-r\sigma)}{\lambda^r}$
Exponential (Exp1) $\theta = \lambda = 1/\beta > 0$ GF: $\alpha_1 = 1,\ \alpha_2 \to \infty$	$\dfrac{\exp(-x/\beta)}{\beta} = \lambda\exp(-\lambda x)$	$1 - \exp(-\lambda x)$	$\beta^r = \lambda^{-r}$
Y= mixture of X with a point mass at x=0. Y= 0 with probability M Y= X with probability 1-M	$f_Y(y) = (1-M)\, f_X(x)$ for $x \neq 0$	*=0 for y < 0* *=M+(1-M)F_X(x)* *for y$0.*	*E(Y^r)=(1-M)E(X^r)*

Notes: $\Gamma(\alpha) = \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$, $B(\alpha_1, \alpha_2) = \Gamma(\alpha_1)\Gamma(\alpha_2)/\Gamma(\alpha_1+\alpha_2)$. "₁ >0 and "₂>0 are one-half the numerator and denominator degrees of freedom. : and F are location and scale parameters for $\log(F)$. If F is an F variate with 2"₁, and 2"₂ degrees of freedom, then the corresponding generalized (power transformed) variate is GF=exp(: +FlogF)=e: F^F. Formulae for moments are valid as long as "₁, "₂, F, "+rF, and " ! rF are all positive. For X a generalized F or any of its special cases, an inverse random variable is obtained via $Y = 1/X$, with properties indicated by the row for Y=1/X. This table was prepared by Lawrence Myers of RTI and is based primarily on Prentice (1975) and Johnson and Kotz (1970).